Graph Based Quantum Error-Correcting Codes

Quantum error-correcting codes (QECC) are necessary to overcome decoherence in quantum computing and also essential in quantum communication. Calderbank, Rains, Shor, and Sloane in their seminal work [1] showed that finding binary quantum error-correcting codes (qubit codes) is equivalent to finding self-orthogonal additive codes over the finite field $\mathbb{F}_4 := \{0, 1, \omega, \overline{\omega} = \omega^2 = 1 + \omega\}$. A code C is called additive if it is closed under addition but not necessarily under multiplication by the elements of $\mathbb{F}_4$. The *trace Hermitian inner product* of $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{F}_4^n$ is given by $\mathbf{x} * \mathbf{y} = \sum_{j=1}^{n} \left(x_j y_j^2 + x_j^2 y_j \right)$. Given an additive code C, its *symplectic dual code* C^* is $C^* = \{\mathbf{x} \in \mathbb{F}_4^n : \mathbf{x} * \mathbf{c} = 0 \text{ for all } \mathbf{c} \in C\}$ and C is said to be *(symplectic) self-dual* if $C = C^*$.

In the literature [3, 4, 6], majority of the best known zero-dimensional qubit codes were constructed from circulant graph based techniques. These types of codes were computationally straightforward to implement and provided better code parameters.
During the 2023 TADM-REU, the Coding Theory Group (CTG) studied multidimensional circulant (MDC) graphs, a generalization of circulant graphs to multiple coordinates [5]. By using the multidimensional construction, they were able to obtain two new record breaking qubit codes. In addition to obtaining isomorphism properties of MDC graphs, the REU group proved that an adjacency matrix of a MDC graph has a nested block circulant structure [7].

In 2024, the CTG investigated generalized Toeplitz graphs and corresponding QECCs [2]. The group were able to construct two new qubit codes with parameter $[[66, 0, 17]]_2$ and $[[92, 0, 22]]$.

During the 2025 REU, we will explore other generalizations of circulant graphs. Projects in this area will require the use of Magma computational algebra system (CAS). The mentor will introduce the Magma CAS and an introduction to coding theory before the start of the TADM-REU. A background in linear algebra and discrete mathematics will be useful.

References

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